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Complexities Due to Sluggish Expansion of Backstop Technologies

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This paper considers an economy using a technology that adds to a stock of pollution. Examples that come to mind are SO_2-emissions from burning coal accumulating in the soil and CO_2-emissions from fossil-energy use which are retained in the atmosphere. The stock of pollutants is subject to natural decay, albeit not necessarily of the simple often assumed linear type. In addition, a clean or so-called backstop technology is available that requires costly investments but is characterized by low variable costs (e.g., solar energy or wind power). The costly investments imply a slow build-up of the capacity of the backstop. On the modelling side, this is an essential extension of most of the literature that considers the unrealistic case where a backstop is instantaneously available. The second extension the present paper makes is to consider not only the planning problem but also the competitive outcomes. One of the interesting results is that stable limit cycles may characterize the socially optimal long-run outcome as well as the competitive equilibrium. In a competitive equilibrium pollution-control policy is not necessarily optimal in the sense of corresponding with the social optimum. Although cycling can occur in a competitive equilibrium, just as in the social optimum, relaxation of the control increases the set of parameter values for which complex and unstable behavior arises.

Keywords: pollution, backstop, limit cycles.

JEL classification: Q2, D6.

1 Introduction

This paper considers an economy using a technology which causes pollution emissions to accumulate (e.g., SO_2 accumulates in the soil, CO_2 in the atmosphere). The pollution stock is subject to natural decay, albeit not necessarily of the simple linear type. In addition, a clean or so-called backstop technology is available that requires costly investment, but can be exploited at low variable costs (say solar energy, wind power, etc.). The costly investments lead to a slow build-up of the backstop capacity. On the modelling side, this offers an essential extension of most of the literature that considers the unrealistic case where a backstop is instantaneously available. Indeed, the obvious fact that all conceivable backstop technologies (say nuclear, solar, renewables) can impossibly overtake such large markets as the world energy market from one day to another, is somewhat overlooked in the literature on backstops (for an exception, see Wirl, 1991). The fact that after decades of talking about backstops, substantial breakthroughs have been achieved recently with fuel cells adds a topical dimension to this investigation. Indeed, DaimlerChrysler will introduce fuel-cell buses this year and plans to sell fuel-cell cars in Europe and America by 2004 (see, *The Economist* July 24th, 1999). This paper provides a theoretical analysis of the introduction of such a backstop, including a complete stability analysis for the social optimum and a competitive equilibrium. One of the interesting results of this framework is that stable limit cycles may characterize the socially optimal long-run outcome. In addition, the paper investigates a competitive equilibrium if the externalities of the dirty good are optimally internalized or not.

2 The Model

The following model is a straightforward amendment of the model studied by Toman and Withagen (2000) to allow for a sluggish build-up of backstop capacities. Environmental pollution (P) or degradation of nature increases by the amount of pollution emitted to the environment, which is assumed to be proportional to the consumption of the dirty product x, minus natural decay, $A(P)$:

$$\dot{P}(t) = x(t) - A(P(t)), \quad P(0) = P_0, \text{ given} . \tag{1}$$

The way the stock of pollution evolves over time can be modelled in several ways. It has been quite common in the early literature to assume a constant exponential rate of decay (see, e.g., Foster, 1975). Then the rate of pollution decay is a linear increasing function of the pollution stock. This is a reasonable specification if one has in mind the decay or the dispersion in the environment of such substances as radioactive materials or greenhouse gases. However, this specification has been criticized by many authors such as Comolli (1977), Dasgupta (1982), Pethig (1993), and Toman and Withagen (2000), based on the observation that high-pollution

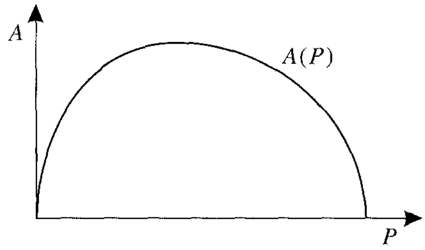

Fig. 1. The decay function

levels may destroy the environment's self-purification processes. A classical reference in the biological literature is Holling (1973). Holling gives several examples where the nutrient enrichment of lakes has caused new equilibria from which the lake cannot recover. An alternative formulation would then include the feature that at sufficiently high levels of pollution the rate of decay is reduced to zero. This is what we have in mind here. We choose a decay function with decay increasing initially, i.e., for small pollution stocks (the parabolic shape in the increasing part can be seen as a generalization of the linear depreciation of the standard approach), and with decay decreasing at larger pollution stocks ($A' > 0$ and $A' < 0$). In fact, decay is inverted U-shaped and concave, $A'' < 0$. See Fig. 1 for a graphical illustration of the decay function. The familiar logistic function provides an arithmetical example.

We consider a partial-equilibrium framework where consumers have utility U from consumption and are indifferent with respect to the origin of the consumer good: the dirty good x or the backstop product B. It is assumed that the instantaneous utility function U is strictly concave and strictly increasing and that $U'(0) = \infty$ and $U'(\infty) = 0$. These assumptions ensure an interior solution. The marginal willingness to pay, U', describes the inverse demand function, i.e., the market-clearing price given aggregate supplies. The costs of producing the dirty good, $C(x)$, are assumed to be increasing and convex in output x, $C' > 0$, $C'' \geq 0$. Finally, existing pollution has social cost D that is strictly increasing and convex. We do not treat the environment as providing raw materials or anything like that.

The extension we propose is to incorporate a backstop capacity that provides a clean substitute, free of variable costs and that is characterized by a sluggish build-up. The assumption of zero variable costs ensures that the installed capacity of the backstop technology is always utilized and thus consumed. Consumption from the backstop can therefore be identified with the backstop capacity B. Of course, zero variable costs are not essential.

They just have to be lower than the costs of the dirty product and the technology itself must be economically viable. The sluggish build-up is due to the fact that the backstop is produced from a capital stock, say fusion or wind power plants, photovoltaic cells, etc., which cannot be implemented at once for various reasons such as adjustment costs. Indeed, the obvious fact that all conceivable backstop technologies can impossibly overtake from one day to another such large markets as the world energy market is somewhat overlooked in the literature on backstops. The accumulation of the backstop capacity is described as follows:

$$\dot{B}(t) = y(t) - \delta B(t), \quad B(0) = 0 . \tag{2}$$

Here y is the investment in the backstop and δ is the constant rate of depreciation. Investment y in the backstop technology has costs I that are strictly increasing and convex.

In the following, we will study the intertemporal evolution of both states, backstop technology and pollution, under two different institutional arrangements: in a social optimum and in a competitive equilibrium.

3 Social Optimum

We start with an analysis of the social optimum. The objective is to determine optimal production of the dirty commodity and investments in the backstop in order to maximize the aggregate net present value of social welfare:

$$\max_{\{x(t),y(t)\}} \int_0^\infty \mathrm{e}^{-rt}\big[U(x(t) + B(t)) - D(P(t)) - C(x(t)) - I(y(t))\big]\,\mathrm{d}t ,$$

subject to (1) and (2). The discount rate is denoted by r.

Current social welfare consists of utility U from consumption, minus the external costs, D, due to existing pollution, minus the costs of producing the dirty good, C, and minus investment expenditures I. In this section we assume that the latter costs depend on production only. This does not exclude the (likely) possibility that production costs increase with pollution (e.g., requiring dams, filters, and other largely fixed-costs elements), because these (additive) costs can be integrated in D, but it excludes that pollution increases the marginal costs of producing the dirty good. The reason for this simplification is purely arithmetical because retaining the pollution in the production costs complicates matters considerably, without adding further insights. To interpret the particular separability of the welfare function one can assume that the function U reflects the monetary revenues of selling the consumer commodity on, e.g., the world market, whereas environmental damage D is also expressed in monetary values.

In order to solve the optimal-control problem we define the current-value Hamiltonian (omitting the time argument t)

$$\begin{aligned} H(x, y, P, B, \lambda, \mu) = U(x + B) - D(P) - C(x) - I(y) \\ + \lambda[x - A(P)] + \mu[y - \delta B] \,. \end{aligned}$$

Note that the Hamiltonian is jointly concave in states and controls, due to the concave objective and given that pollution carries a negative shadow price λ. Therefore, the first-order conditions together with the transversality conditions are sufficient for an optimal program. The transversality conditions are satisfied if the states and co-states converge to a finite steady state or remain bounded, as in the case of limit cycles.

The first-order conditions for an interior solution are the Hamiltonian-maximizing conditions,

$$\begin{aligned} H_x = 0 &: \ U' - C' + \lambda = 0 \,, \\ H_y = 0 &: \ -I' + \mu = 0 \,, \end{aligned}$$

and the differential equations for the shadow prices of the stock of pollution λ and of the backstop capacity μ, given by:

$$\begin{aligned} \dot{\lambda} &= (r + A')\lambda + D' \,, \\ \dot{\mu} &= (r + \delta)\mu - U' \,. \end{aligned}$$

If we assume that interior controls exist, we can write $x = X(B, \lambda)$, from the first necessary condition, and $y = Y(\mu)$, from the second necessary condition, with the derivatives, $X_B = U''/(C'' - U'')$, $X_\lambda = 1/(C'' - U'')$, $Y_\mu = 1/I''$, given by the implicit-function theorem. In the sequel it is assumed that there exist steady states where the controls are indeed interior, e.g., due to the above mentioned Inada conditions. With regard to the points we wish to make this is not restrictive. We shall also provide some examples where this is straightforward to establish. Substitution of the optimal controls into state and co-state equations yields the following canonical equation system:

$$\begin{aligned} \dot{P} &= X(B, \lambda) - A(P) \,, \\ \dot{B} &= Y(\mu) - \delta B \,, \end{aligned}$$

$$\dot{\lambda} = (r + A'(P))\lambda + D'(P) ,$$
$$\dot{\mu} = (r + \delta)\mu - U'(X(B, \lambda) + B) .$$

From this system it follows that in the steady state $r + A' > 0$, because the co-state of pollution is negative.

We now state

Proposition 1: Suppose that in the stationary state $A'(P) > 0$. Then the optimal steady state is asymptotically locally stable.

The result follows from applying standard sufficiency criteria (such as developed by Brock, Scheinkmann, and others). The formal proof is relegated to Appendix 1, where however other more direct techniques are employed. The economic consequence of the result is that sufficient environmental concern, implying that pollution is below what "nature" could digest, i.e., P such that $A'(P) > 0$, does not only lower stationary pollution but also ensures stability. However, other cases might occur as well.

Proposition 2: If stationary pollution is "large" so that $A'(P) < 0$, then this steady state may be asymptotically locally stable but it may also be unstable and, in particular, there may exist stable limit cycles.

The existence of limit cycles means that the build-up of the backstop capacity in order to reduce pollution and thus to lower the pressure on the environment, is followed by an increased consumption of the dirty product as the backstop capacity depreciates. Yet at higher pollution levels, the backstop is again pushed back into the market, and so on forever.

The reason for the existence of limit cycles is that the framework described by the differential equations (1)–(2) includes one of the routes to limit cycles in strictly concave models addressed in Wirl (1996). More precisely, Wirl (1996) shows that growth – the derivative of a state differential equation with respect to the corresponding state is positive, but less than the rate of discount – is a pathway to obtain limit cycles. In the case at hand the condition of growth amounts to $r > \partial\dot{P}/\partial P = -A'(P) > 0$. Although that result is strictly speaking not applicable since we have two instead of the scalar control assumed in Wirl (1996), this condition extends apparently to multiple controls. In the case at hand the inequality is also a necessary condition for a Hopf bifurcation, which ensures the existence of stable (and generic) limit cycles. Furthermore it is worth noting that while "growth" is destabilizing, the second dynamics $r > \partial\dot{B}/\partial B = -\delta$ (< 0) is stabilizing. Hence $\delta > 0$ restricts the domain of complexities even for $A' < 0$ up to the point of ensuring overall stability if the rate of depreciation is sufficiently large. Or, the other way around, long-lasting backstop (i.e., small δ) capacities are suitable to yield limit cycles. However, the restriction imposed by δ is implicit rather than explicit for the social optimum. It is discussed in detail in Appendix 2.

The Hopf bifurcation theorem, existence of stable limit cycles, requires that three properties hold:

i. There exists a pair of purely imaginary eigenvalues for a proper choice of the parameter that is varied (called the critical value or bifurcation point).
ii. The derivative of the real part of the eigenvalues with respect to the parameter is different from zero. Hence the critical value of the parameter separates the domains where the linearized system is stable (possibly restricted to a "stable" manifold) from the domain of locally unstable spirals.
iii. The coefficient of a quadratic term of the so-called normal form is negative (see Guckenheimer and Holmes, 1983).

Conditions (i) and (ii) ensure the existence of a limit cycle. Yet if condition (iii) is violated the cycle is unstable. That is, such a cycle repels all motions starting arbitrarily close to the cycle (and within the stable manifold): motions starting inside the cycle converge to the steady states, those starting outside the cycle either diverge or converge to another steady state (if existing). Stability of the cycle requires a "supercritical" bifurcation.

The geometric intuition of the theorem is straightforward: at the point of the bifurcation, the steady state of the linearized system becomes a centre, but the nonlinear system remains stable because of the negative quadratic terms. For parameter values slightly beyond the critical value, there are two opposing forces. First the negative quadratic term addressed in (iii) dominates the linear terms, at least sufficiently off the equilibrium. Second, close to the equilibrium this quadratic term is irrelevant so that the linear terms, with positive real parts, lead to locally "exploding" spirals near the equilibrium. The limit cycle arises from balancing these two forces and constitutes the attractor of this system. If, on the other hand, the quadratic term is positive, a cycle requires that the linear terms provide the stabilizing elements so that the steady state is locally stable and the cycle becomes repelling. For further details see Guckenheimer and Holmes (1983). In the following we will concentrate on condition (i). We will verify the other conditions numerically.

In the context of our model, the growth condition mentioned requires that $A' < 0$, meaning a relatively large socially optimal pollution stock. Given this condition, very simple examples, with high discount rates and/ or highly convex investment costs, allow for a Hopf bifurcation and hence for stable limit cycles. We consider an example with linear external costs $D(P) = dP$, linear production costs $C(x) = cx$, quadratic instantaneous utility of consumption, $U(z) = z - \frac{1}{2}z^2$, so that demand is linear with a maximal marginal willingness to pay of 1\$ and maximum demand of 1 unit, linear-quadratic investment costs $I(y) = ay + \frac{1}{2}by^2$, and logistic decay $A(P) = P(1 - P)$. For graphical purposes we choose $a = 0.1$, $c = 0.2$, $d = 0.3$, $\delta = 0.05$, and $r = 1.8$. Thus the average lifetime of a backstop plant is 20 years. We use the parameter b, which determines the convexity of the investment costs, as the bifurcation parameter. This approach leads to a pair of purely imaginary eigenvalues of the Jacobian at $b = b^{\text{crit}} = 6.56596...$, where the derivative of the real part does not vanish and the bifurcation is supercritical, which was verified numerically using LOCBIF (see Khibnik et al., 1992). Therefore, limit cycles characterize the optimal policy in a local, one-sided surrounding of $b > 6.56596...$ In Fig. 2, we vary the adjustment cost parameter b as indicated for the reported Hopf bifurcation which yields the interesting result, that "intermediate" and high values induce limit cycles or instability, but low and very high values for b imply stability. Of course, the instability results only upon entering the domain $P > 1/2 = \arg\max A(P)$ so that $A' < 0$.

It could be argued that in order to obtain limit cycles an unrealistically high discount rate is needed. However, this barrier could be reduced by simultaneously reducing the rate of depreciation of the backstop technology δ. Moreover, although the parameters might not allow for exactly stable limit cycles, there might occur damped cycles which are difficult to discern from stable limit cycles in practice. Moreover, in practice the system will always be distorted by additional factors.

Figure 2 shows the steady states in the state plane (P, B) and the associated stability properties for variations of the discount rate with all other parameters as above and b at the critical value for $r = 1.8$. These variations lead to stable Hopf bifurcation too and highlight at the same time the nonmonotonic dependence of the steady states on the rate of the discount.

Setting $\delta = 0$ and assuming $I(0) = 0$ implies the same steady state as reported in Toman and Withagen (2000). Yet the inclusion of adjustment

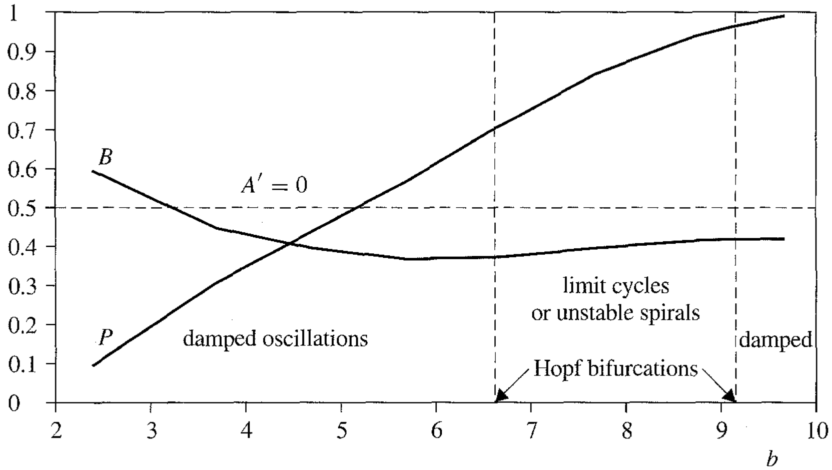

Fig. 2. Steady states, stability properties and Hopf bifurcations versus b

costs can fundamentally alter the stability properties. More precisely, adjustment costs can destabilize an otherwise stable equilibrium, but cannot stabilize an otherwise unstable equilibrium (see Feichtinger et al., 1994). Therefore, considering very long lasting equipment can generate Hopf bifurcations at much lower discount rates. For example, setting $a = 0.95$, $c = 0.3$, $d = 0.001$, $\delta = 0.001$, and $r = 0.3$ yields a Hopf bifurcation at $b = 132.5144$ (with the steady states $B = 0.457$ and $p = 0.517$, thus y close to its "maximum" of 0.25). The example of the decay function used in the next section, $A(P) = P\left(1 - \sqrt[3]{P}\right)$, enlarges the critical domain, which simplifies the location of bifurcation, e.g., $a = 0.3$, $c = 0.6$, $d = 0.001$, $\delta = 0.001$, and $r = 0.2$ yields stable limit cycles for $b = 73.28402$ around $B = 0.29$ and $P = 0.516$.

4 Competitive Equilibria

In this section, we study the perfect-foresight competitive equilibrium where the externality is either not at all internalized or not optimally internalized by governmental environmental policy. The representative firm faces the following problem:

$$\max_{\{x(t),y(t)\}} \int_0^{\infty} \mathrm{e}^{-rt}\big[p(t)[x(t) + B(t))] - C(x(t)) \\ - \tau(t)x(t) - I(y(t))\big]\,\mathrm{d}t\ ,$$

subject to $\dot{B} = y - \delta B$, $B(0) = 0$. So, the firm decides how much to pollute, $x(t)$, and how much to expand the backstop capacities, $y(t)$, taking the market price, $p(t)$, and the pollution stock, $P(t)$, as given. The objective is to maximize the aggregate net present value of profits consisting of the revenues, $p[x + B]$, minus the production costs, C, and investment costs, I, and minus the pollution taxes, τx, where τ denotes the unit tax on the production of the dirty commodity, which is exogenous to the firm. Each individual firm neglects environmental damage, and pollution does not enter the private cost function either. Firm-specific feedback of pollution by means of the tax rate is now essential in contrast to the social optimum where such feedbacks were included in D. In the absence of such a feedback, the tragedy of the commons arises, ultimately destroying the environment's inherent abatement capabilities. The consequences of such irreversibilities are analyzed by Tahvonen and Withagen (1996) in the context of a planning problem. The irreversibility creates a nonconcavity for the planning problem that would be irrelevant in our competitive setting, because we are working in a decentralized economy.

The Hamiltonian of the system reads

$$H(x, y, B, \nu) = p[x + B] - C(x) - \tau x - I(y) + \nu[y - \delta B] .$$

Assuming an interior solution we find as a first set of necessary conditions the Hamiltonian-maximizing conditions:

$$H_x = p - C' - \tau = 0 ,$$
$$H_y = -I' + \nu = 0 .$$

The optimal controls, i.e., production of the dirty product and investment in the backstop, are now implicitly defined by $x^{\mathrm{c}} = X^{\mathrm{c}}(B; \tau)$ and $y^{\mathrm{c}} = Y^{\mathrm{c}}(\nu)$. Applying the implicit-function theorem and supposing market clearing at any instant of time

$$p(t) = U'(x(t) + B(t))$$

yields the derivatives:

$$X^{\mathrm{c}}_B = -U''/(U'' - C''), \quad X^{\mathrm{c}}_\tau = 1/(U'' - C''), \quad \text{and} \quad Y^{\mathrm{c}}_\nu = 1/I'' .$$

Another necessary condition is the differential equation for the single adjoint variable of the representative firm's backstop capacity, denoted by ν, to differentiate from the socially optimal solution where we used μ:

$$\dot{\nu} = (r + \delta)\nu - p .$$

Moreover, Eq. (1) describing the evolution of the stock of pollution must hold. Although the competitive firms have no control over the stock of pollution, they perfectly anticipate the evolution of pollution (rational expectations). Therefore, the competitive equilibrium is described by the following system of differential equations:

$$\dot{B} = Y^{\mathrm{c}}(\nu) - \delta B \ ,$$
$$\dot{\nu} = (r + \delta)\nu - U'(X^{\mathrm{c}}(B; \tau) + B) \ ,$$
$$\dot{P} = X^{\mathrm{c}}(B; \tau) - A(P) \ .$$

The following proposition identifies the relationship between a competitive equilibrium and the social optimum.

Proposition 3: Suppose the tax rate is set such that it solves $\tau = -\lambda$, where λ is the co-state of the pollution stock in the social optimum, and satisfies $\dot{\lambda} = (r + A')\lambda + D'$ and a transversality condition. Then the competitive equilibrium is identical to the social optimum.

Proof: Addition of this differential equation for $\dot{\lambda}$ to the competitive equilibrium and replacing τ by $-\lambda$ yields the same system as in Sect. 3 except for a different labelling of ν instead of μ. □

Proposition 3 leads immediately to the corollary that competitive equilibria allow for limit cycles, too. Nevertheless we study in the following also a competitive equilibrium where internalization is possibly sub-optimal, so that the tax rate is not equal to the negative of the shadow price of pollution.

Proposition 4: Suppose that the steady state is in the domain of $A' > 0$. Then the steady state is asymptotically locally stable.

This proposition is completely analogous to Proposition 1. The next two propositions investigate the domain $A' < 0$.

Proposition 5: Suppose that the tax rate τ is constant over time. Then a steady state in the domain of $A' < 0$ is (generically) unstable, i.e., only one eigenvalue is negative so that the stability is restricted to a one-dimensional manifold of the initial conditions in the (P, B)-plane.

The instability addressed in Proposition 5 is usually associated with multiple equilibria with the consequence that applying even the long-run optimal tax is insufficient, if initial pollution is large. Note that the instability addressed in Proposition 5 can occur in all other cases considered in this paper. This may be surprising to some of the readers given the strict and joint concavity of our model, since in the literature most of these kinds of instabilities are associated with (local) convexities.

An immediate consequence of Proposition 5 is that we have to extend the analysis so as to allow for a tax that is increasing in pollution, to get complex solutions. The reason is that the kind of instability addressed in Proposition 5 excludes local instabilities of the kind required for a Hopf bifurcation. In a four-dimensional system of states and co-states both uniqueness of the optimal policy and saddlepoint stability require a stable two-dimensional manifold, corresponding with two eigenvalues that are either negative or have negative real parts. One negative eigenvalue, as in the case of one state variable, reduces the stability to a one-dimensional manifold in the state space, so that the generic outcome is a local instability. Therefore, saddlepoint stability in this sense is equivalent to two eigenvalues being negative or having negative real parts.

The relation $\tau = \tau(P)$ may cover to some extent not only the tax but also other adversities to the firm related to pollution, such as the increase of marginal costs of producing the dirty good. Substitution of this modification into the above differential-equation system characterizes the competitive equilibrium facing a state contingent, instead of a constant or just time-dependent tax rate.

Proposition 6: Instabilities and limit cycles can characterize a competitive equilibrium with a stationary pollution such that $A' < 0$. However, the domain for complexities in competitive equilibria is not restricted by the discount rate of the (representative) firm, i.e., $r + A' < 0$ is a possible long-run outcome under competition. Yet depreciation restricts the domain of possible complexities of limit cycles to sufficiently long-lasting capacities, $\delta + A' < 0$, so that for $\delta = 0$ the entire domain $A' < 0$ permits complexities.

The reason for this restriction is the stability inherent to the accumulation of the backstop capacities. In contrast to the social optimum an explicit bound can be given concerning the domain of complex solutions. An example establishing the claim of limit cycles in Proposition 6 borrows from the social-planning example the specifications of the surplus, $U(z) = z - \frac{1}{2}z^2$, and of the investment costs, $I(y) = ay + \frac{1}{2}by^2$, but assumes quadratic production costs $C(x) = \frac{1}{2}cx^2$, decay slightly different from the logistic one, $A(P) = P\left(1 - \sqrt[3]{P}\right)$, and firm-specific taxes/costs linear in pollution, $\tau(P) = sP$. A theoretical discussion and details concerning this example are given in the Appendix 2. Setting the parameters $a = 0.1$, $c = 0.05$, $r = 0.3$, $\delta = 0.05$, and $s = 0.1$ yields a supercritical Hopf bifurcation for variations in b at $b = 3.870334$ (i.e., a pair of purely imaginary eigenvalues satisfying the conditions about a nonzero crossing velocity according to the calculations performed with LOCBIF). The resulting steady state of large pollution exceeds pollution feasible for a socially optimal programme and thus highlights that the domain of complex policies is enlarged for competitive outcomes.

Figure 3 compares the stability properties of the competitive solution with the social optimum. This comparison uses the specifications and the parameters ($a = 0.1$, $c = 0.05$, $\delta = 0.05$, and $s = 0.1$) of the above-mentioned bifurcation for the competitive, intertemporal equilibrium. The external costs are $D(P) = dP$ with $d = 0.005$. The figure considers variations in the discount rate r instead of b. More precisely, Fig. 3 shows the steady states and the corresponding stability properties for the intertemporal, competitive equilibrium in Fig. 3a and of the social optimum in Fig. 3b versus the discount rate. Using the linear damage function and a sufficiently small damage parameter leads to comparable levels of pollution. Although only competition induces limit cycles while the system remains

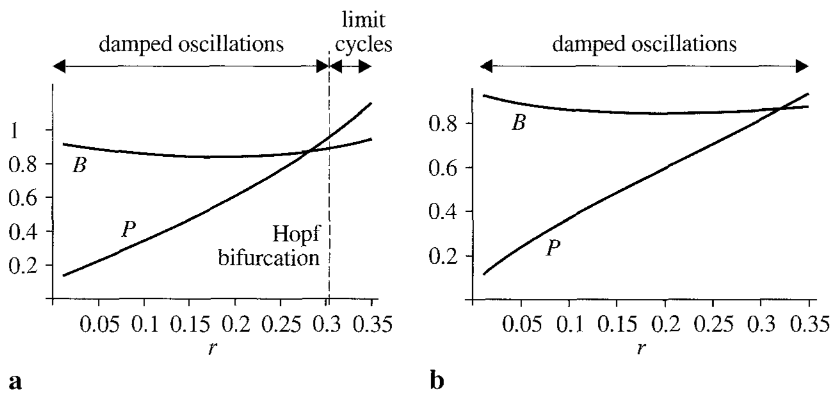

Fig. 3a, b. Steady states and the associated stability properties versus the discount rate r. **a** Competition, $s = 0.1$, **b** social optimum, $d = 0.005$

saddle-point stable in the social optimum, damped oscillations are socially optimal over the entire considered domain of discount rates.

5 Concluding Remarks

It has been shown that with sluggish build-up of a backstop technology complex behavior of pollution and backstop capacity can arise, in a social optimum as well as in a competitive equilibrium. A remarkable result is that in a competitive equilibrium with nonoptimal taxation the scope for such behavior is larger. This fact has been established by means of a numerical example. Figure 4 summarizes these general findings.

The basic reason for the occurrence of limit cycles is that the consideration of negative marginal decay introduces "growth," which is a pathway to Hopf bifurcation and thus for limit cycles in strictly concave dynamic optimization models. Local convexities substantially simplify the derivation of limit cycles from the first-order conditions, but these conditions are not sufficient anymore. Wirl (1996) shows this for planning problems and Wirl (1997) for competitive equilibria. The reason for the differences between the social optimum and the competitive equilibrium is that the accumulation of pollution beyond the stock where $r + A' = 0$ is always socially suboptimal, but it is a feasible outcome under competition if the externality is insufficiently internalized. Furthermore, the relation between

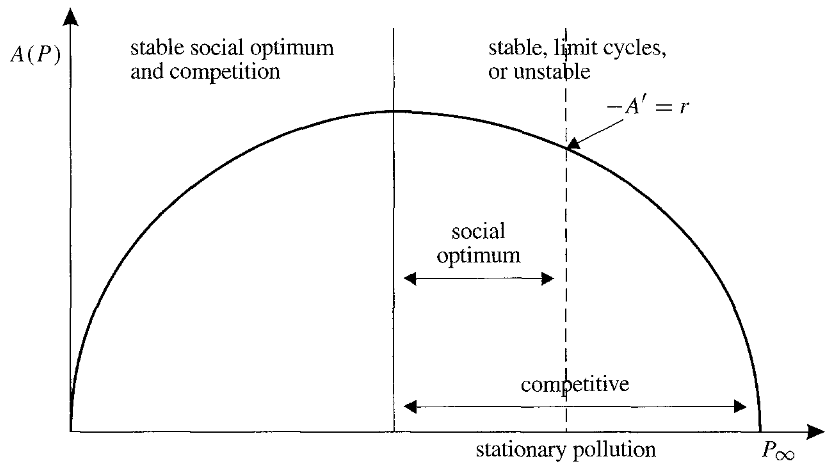

Fig. 4. Comparison of the stability properties of social planning and competition

social optimum, competition, and taxation is subtle. First, a steady state of a competitive economy, where $A' < 0$ and the firms face a constant tax, is unstable. Hence, even a stable social optimum within the critical domain $r > -A'$ and imposing the tax equal to the long-run optimum is insufficient for stability of the competitive economy. This result in turn implies that the tax must depend (positively) on the stock of pollution to allow for cyclical intertemporal equilibria.

This paper has two different messages. First, growth dynamics, which appear to be a characteristic feature of environmental and biological resilience, allow for the puzzling feature of limit cycles as optimal strategies. In our opinion, this result, which has little intuitive appeal at first sight, is of considerable theoretical interest since our science (called "dismal") lacks paradoxical results. Of course, the domain where these kinds of strategies are indeed optimal can be rather narrow. Therefore, the second conclusion from this paper is more relevant for applied policy making: neglecting adjustment costs and thereby reduction to lower-dimensional systems may substantially underestimate the complexity of the optimal policies in environmental systems. More precisely, some kinds of oscillations (possibly damped or even more complex in higher-order systems coupled with growth elements, predator–prey interactions, etc.) are optimal for the result of competitive interactions. This was demonstrated here for the introduction of backstops: the apparently innocuous and highly plausible introduction of sluggish capacity expansion has not necessarily the smoothing effect expected to result from this modification but can lead to oscillations (damped, persistent, exploding). This is particularly the case as long as the external costs are small.

Although our paper has introduced dynamics related to backstop technologies, it stops half way. A major claim concerning the on-going subsidizing of backstop technologies is that their efficiency will improve over time and enhances learning by doing. This leads to two opposing forces. Learning by doing fosters a large output of new backstop capacities, yet such a rapid expansion excludes the use of even more efficient equipment in the future providing another reason for a sluggish build up of capacities. How much this affects the outcome is left for future research.

Appendix

1 Stability Analysis in the Social Optimum

The stability properties of the system introduced in Sect. 3 may be studied applying the global asymptotic stability criteria developed in the seventies by Brock, Malliaris, Rockefeller, Scheinkman, and others, which are well summarized in the book of Brock and Malliaris (1989). However, we opt for a local stability analysis. This requires the calculation of the eigenvalues of the Jacobian J of the canonical equations system discussed in Sect. 3, evaluated at a steady state. Applying Dockner (1985), the eigenvalues E_i ($i = 1, 2, 3, 4$) are given by

$$E_i = r/2 \pm \sqrt{(r/2)^2 - K/2 \pm (1/2)\sqrt{K^2 - 4\det(J)}}, \quad i = 1, 2, 3, 4 .$$

Here K is defined as follows (see Dockner, 1985, p. 96).

$$K = \begin{vmatrix} \frac{\partial \dot{P}}{\partial P} & \frac{\partial \dot{P}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial P} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{B}}{\partial B} & \frac{\partial \dot{B}}{\partial \mu} \\ \frac{\partial \dot{\mu}}{\partial B} & \frac{\partial \dot{\mu}}{\partial \mu} \end{vmatrix} + 2 \begin{vmatrix} \frac{\partial \dot{P}}{\partial B} & \frac{\partial \dot{P}}{\partial \mu} \\ \frac{\partial \dot{\lambda}}{\partial B} & \frac{\partial \dot{\lambda}}{\partial \mu} \end{vmatrix} .$$

The formula allows for a complete characterization of the local dynamics of the linearized system and provides the ideal test recommended in Brock and Malliaris (1989, bottom of p. 148). For the general case of Sect. 3 we have:

$$J = \begin{pmatrix} -A' & \frac{U''}{C'' - U''} & \frac{1}{C'' - U''} & 0 \\ 0 & -\delta' & 0 & 1/I'' \\ A''\lambda + D'' & 0 & r + A' & 0 \\ 0 & -\frac{U''C''}{C'' - U''} & -\frac{U''}{C'' - U''} & r + \delta \end{pmatrix} .$$

Calculation of the coefficients in Dockner's formula yields for K:

$$K = -A'[r + A'] - \delta[r + \delta] - \frac{A''\lambda + D''}{C'' - U''} + \frac{C''U''}{I''[C'' - U'']} .$$

The three final terms of this expression are negative, due to the assumed second-order derivatives and to the fact that the stationary shadow price of pollution, $\lambda = -D'/(r + A')$, is negative. Only $-A'(r + A')$ is of ambiguous sign. The determinant equals:

$$\det(J) = \delta A'[r+\delta][r+A'] + \frac{\delta[r+\delta][\lambda A'' + D'']}{C'' - U''} - \frac{U''[\lambda A'' + D'' + A'C''[r+A']]}{I''[C'' - U'']} .$$

According to the above calculations, low pollution, i.e., $A' > 0$, implies $K < 0$ and $\det(J) > 0$. In view of Dockner's formula these inequalities are sufficient for J to have two (and only two) eigenvalues, which are either negative or have negative real parts. That implies saddlepoint stability, albeit that damped oscillation may be optimal, since the eigenvalues of the stable manifold can be complex (as shown in the example in the main text). This proves Proposition 1. □

We show next that proposition 2 holds. First, $A' < 0$ still allows for stability. This follows directly from the above calculations because $A' < 0$ is compatible with $K < 0$ and $\det(J) > 0$, which in turn are sufficient for saddlepoint stability. The example in Fig. 2 shows that even local monotonicity is possible for $A' < 0$. The existence of limit cycles according to the Hopf bifurcation theorem requires inter alia the existence of a pair of purely imaginary eigenvalues. Assume in Dockner's formula the existence of a pair of purely imaginary eigenvalues, $E_{34} = \pm iw$, move $r/2$ to the left and square,

$$r^2/4 \pm riw - w^2 = (r/2)^2 - K/2 \pm (1/2)\sqrt{K^2 - 4\det(J)} .$$

Cancelling $r^2/4$ on both sides and equating coefficients of real

$$-w^2 = -K/2 ,$$

and imaginary parts

$$rw = (1/2)\sqrt{K^2 - 4\det(J)} ,$$

squaring and using $w^2 = K/2 > 0$ implies

$$\det(J) = (K/2)^2 + r^2 K/2, \quad K > 0 .$$

This in turn requires that the determinant is positive, too.

According to the above calculation, $K > 0$ requires $A' < 0$. Ironically enough, this negative derivative introduces "growth," which according to Wirl (1996) is a pathway for limit cycles. As mentioned earlier, Wirl (1996) shows that growth, i.e., $r > \partial \dot{P}/\partial P = -A' > 0$, is a necessary condition for Hopf bifurcation in strictly concave dynamic optimization models and of the possible pathways only this one is present. Although his theorem is, strictly speaking, not applicable because of its restriction to a single control, the consequence on $K > 0$ is tied to $A' < 0$. However, depreciation of the backstop capacities adds a stabilizing (i.e., negative) factor in K so that $A'[r + A']$ must at least outweigh $\delta[\delta + r]$. Hence, long-lasting investments are helpful for $K > 0$, a prerequisite for complex solutions such as limit cycles.

For the example in Sect. 3, we obtain the optimal controls, $x^* = 1 - B - c + \lambda$, $y^* = [\mu - a]/b$. The canonical equations (retaining $A = P[1 - P]$ and $A' = 1 - 2P$ as short hand) are:

$$\begin{aligned}
\dot{P} &= (1 - B - c + \lambda) - A(P) , \\
\dot{B} &= [\mu - a]/b - \delta B , \\
\dot{\lambda} &= [r + A']\lambda + d , \\
\dot{\mu} &= [r + \delta]\mu + \lambda - c ,
\end{aligned}$$

which yields

$$J = \begin{pmatrix} -A' & -1 & 1 & 0 \\ 0 & -\delta & 0 & 1/b \\ -2\lambda & 0 & r + A' & 0 \\ 0 & 0 & 1 & r + \delta \end{pmatrix} .$$

This system allows for a closed-form analytical solution, of the steady states and even of the critical value of the bifurcation parameter, at least for the parameter b. However, all these expressions are extremely cumbersome so that we report here only the crucial coefficient K:

$$K = -A'[r + A'] + 2\lambda - \delta[r + \delta] .$$

Fairly similar are the results for the example in Sect. 4, albeit that we could not obtain a closed-form solution anymore given the quadratic production costs:

$$\begin{aligned}
\dot{P} &= \frac{1 - B + \lambda}{1 + c} - A(P) , \\
\dot{B} &= \frac{\mu - a}{b} - \delta B ,
\end{aligned}$$

$$\dot{\lambda} = [r + A']\lambda + d\,,$$
$$\dot{\mu} = [r + \delta]\mu - \frac{c[1 - B] - \lambda}{1 + c}\,.$$

This system yields the following Jacobian:

$$J = \begin{pmatrix} -A' & \dfrac{-1}{1+c} & \dfrac{1}{1+c} & 0 \\ 0 & -\delta & 0 & 1/b \\ \lambda A'' & 0 & r + A' & 0 \\ 0 & \dfrac{c}{1+c} & \dfrac{1}{1+c} & r+\delta \end{pmatrix}.$$

The corresponding K is

$$K = -A'[r + A'] - \frac{A''\lambda}{1+c} - \frac{c}{b[1+c]} - \delta[r+\delta]\,.$$

2 *Stability in the Competitive Equilibrium*

Although the following analysis allows for taxes depending on pollution we start with the case where the tax rate is independent of pollution. Then the corresponding Jacobian equals

$$J = \begin{pmatrix} -\delta & 1/I'' & 0 \\ \dfrac{U''C''}{[U'' - C'']} & r+\delta & 0 \\ \dfrac{-U''}{U'' - C''} & 0 & -A' \end{pmatrix}.$$

Again the stability properties can be obtained from calculating the eigenvalues of the Jacobian. The eigenvalues are the roots of the characteristic polynomial:

$$p(e) = e^3 - \mathrm{tr}(J)e^2 + ke - \det(J)\,,$$

where k is the sum of the principal minors of dimension 2 of the Jacobian. It plays a role similar to K in Appendix 1. The calculation of the coefficients of the characteristic polynomial proceeds in several steps:

$$\mathrm{tr}(J) = r - A' ,$$

$$\det(J) = A'\delta(r+\delta) + \frac{U''A'C''}{I''(U''-C'')} ,$$

and

$$k = -\delta[r+\delta] - rA' - \frac{C''U''}{I''[U''-C'']} .$$

Since $A' < 0$ implies $\det(J) < 0$, an instability arises whenever $A' < 0$ and thus the impossibility of limit cycles. This verifies Proposition 5. For $A' > 0$, we have $\det(J) > 0$ as well as $k < 0$. From $\det(J) > 0$ follows that either one eigenvalue, say e_1, is positive and the other two (say e_2 and e_3) are negative, or all three are positive. Since $k = e_3[e_1 + e_2] + e_1 e_2$, $k < 0$ is only possible if both e_2 and e_3 are negative [because they cannot have opposite sign due to the fact that $\det(J) > 0$]. Therefore, the properties $k < 0$ and $\det(J) > 0$ are sufficient for saddlepoint stability. Similarly, $\det(J) > 0$ combined with $\mathrm{tr}(J) < 0$ is sufficient for saddlepoint stability. Therefore, the existence of complex competitive equilibria requires a state-contingent effect, either as a tax or through a feedback on costs. Taking into account $\tau = \tau(P)$ in the system derived in Sect. 4 and applying the chain rule yields the following Jacobian:

$$J = \begin{pmatrix} -\delta & 1/I'' & 0 \\ \dfrac{U''C''}{U''-C''} & r+\delta & \dfrac{-U''\tau'}{U''-C''} \\ \dfrac{-U''}{U''-C''} & 0 & -A' + \dfrac{\tau'}{U''-C''} \end{pmatrix} .$$

We now have

$$\mathrm{tr}(J) = r - A' + \frac{\tau'}{U''-C''} ,$$

$$\det(J) = A'\delta(r+\delta) + \frac{U''(\tau' + A'C'') - \delta\tau' I''(r+\delta)}{I''(U''-C'')} ,$$

and

$$k = -\delta[r+\delta] - rA' - \frac{C''U''}{I''[U''-C'']} + \frac{r\tau'}{U''-C''} .$$

The existence of a pair of purely imaginary eigenvalues as a necessary condition for a Hopf bifurcation requires that $\det(J) = \mathrm{tr}(J)k$ and that all elements of this equation are positive. Therefore, again the sum of the principal minors of dimension 2 must be positive, $k > 0$. Again $A' < 0$ is helpful for a positive trace and a positive k, but can lead to a negative determinant, which implies an instability, yet simultaneously excluding limit cycles. However, $A' < 0$ is not sufficient since $k > 0$ demands at the minimum $\delta + A' < 0$. The reason is similar to the social optimum: $A' < 0$ introduces growth into the externality and thus helps to destabilize the system while $\partial \dot{B}/\partial B = -\delta$ destabilizes the system. Hence instabilities of any kind require that the destabilizing element outweighs the stabilizing element $\delta + A' < 0$. However, in contrast to the social optimum, low discount rates do not constrain the domain of feasible equilibria so that $r + A' < 0$ is feasible.

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